Chapter 2

Sets and Mathematical Induction

In this lecture, we study the notions of sets, relations and functions which are basic tools of discrete mathematics. The concept of a set appears in all mathematical structures.

2.1 Sets

Definition 2.1. A set is defined as the well-defined collection of objects, called the members of elements of the set.

The sets are denoted by the uppercase alphabets such as $A, B, C, \ldots$, whereas the elements of a set are denoted by lowercase letters such as a, b, c, and so on.
If a is an element of set A, then we write it as $a \in A$, which is read as "a belongs to A." Similarly, if a is not an element of A, then it is written as $a \notin A$, read as "a does not belong to A."

Let A be a set containing the elements a, b, and c. Then it is described by listing the elements of the set between braces and the elements are separated by commas. Hence,

$$A = \{a, b, c\}.$$

Remark: It is important to note that the order in which the elements of a set are listed is not important. Therefore, $\{b, a, c\}, \{b, c, a\}, \{a, c, b\}, \{c, a, b\}, \{c, b, a\}$ are the representations of the same set A.

Set Formation

The set can be formed in two ways:
(i) Tabular form of a set, and

(ii) Builder form of a set.

(i) Tabular form of a set

Definition 2.2. If a set is formed by listing its members, then it is called ***tabular form of a set***.

Example 2.1. If set A contians elements 0, 1, 2, 3, then it is expressed as $A = \{0, 1, 2, 3\}$.

(ii) Builder form of a set

Definition 2.3. If a set is defined by the properties that its elements must satisfy, then it is called ***builder form of a set***.

Example 2.2. (i) $A = \{x \mid x \in \mathbb{N}, x \text{ is a multiple of } 5\}$.
(ii) $B = \{x \mid x \text{ is odd number and } x \text{ is less than } 20\}$.

Subset

Definition 2.4. Suppose A and B are any two sets. Then A is called a ***subset*** of B, symbolically, $A \subseteq B$, if and only if all the elements of A are also the elements of the set B.

On the other hand, a set A is not a subset of B, written as $A \nsubseteq B$, if and only if there is at least one element of A that is not in B.

Example 2.3. If $A = \{1, 3, 6\}$ and $B = \{3, 6, 9, 2, 1\}$, then A is the subset of B i.e., $A \subseteq B$.

Since every element in a set A is in A, it follows that any set A is a subset of itself.

Proper Subset

Definition 2.5. Let A and B be sets. Then A is said to be a ***proper subset*** of B, denoted by $A \subset B$, if and only if, every element of A is in B but there is at least one element of B that is not in A.

Example 2.4. The set $A = \{l, m, n\}$ is a proper subset of the set $B = \{j, k, l, m, n, o, p\}$.

Equal Sets

Definition 2.6. Two sets A and B are said to be ***equal***, written as $A = B$, if every element of A is in B and every element of B is in A.

If $A = B$, then $A \subseteq B$ and $B \subseteq A$. Two sets are equal if and only if they have the same elements in it.

Improper Subset

Definition 2.7. If a set A is a subset of set B, and $A = B$, then A is said to be an ***improper subset*** of B.

Example 2.5. If $A = \{a, b, c\}$, and $B = \{a, b, c\}$ then A is improper subset of B since $A = B$.

Remark: Every set is improper subset of itself.

Example 2.6. Let

$$A = \{1, 2, 3, 4, 5\},\ B = \{x \mid x \text{ is a positive integer and } x^2 < 30\}.$$

Is $A = B$?

Solution. We find the tabular form of the set B to check to check if it is equal to the set A. Since x is a positive integer and $x^2 < 30$, it shows that $1^2 = 1, 2^2 = 4, 3^2 = 9$, $4^2 = 16, 5^2 = 25$, but the square of any other positive integer is more than 30. So,

$$B = \{1, 2, 3, 4, 5\} = A$$

Therefore, $A = B$.

Transitive Property of Subsets

If A, B, and C are sets and if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Empty Set

Definition 2.8. A set which contains no element in it is called an ***empty*** or ***null*** or ***void*** set. It is denoted by $\emptyset$ or simply $\{\}$.

Power Set

Definition 2.9. The set of all subsets (proper or not) of a set A, written as $P(A)$, is called the ***power set*** of A.

Example 2.7. If $A = \{a, b, c\}$, then find the power set of A?

Solution. $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.
We note that all the members of $P(A)$ are proper subsets of A except for $\{a, b, c\}$.

Note:
The number of elements in the above set A, denoted by $|A| = 4$.
Number of member of $P(A)$ is $|P(A)| = 8 = 2^3$

Theorem 2.1. *If the set A has n elements i.e., $|A| = n$, then its power set will always have 2^n elements in it, that is*

$$|P(A)| = 2^n.$$

Theorem 2.2. *Let A and B be two sets. If $A \subseteq B$, then $P(A) \subseteq P(B)$.*

Universal Set

Definition 2.10. If we deal with sets all of which are subsets of a set U, then this set U is called a ***universal set*** or a ***universe of discourse*** or a ***universe***.

Union of Sets

Definition 2.11. Let A and B be subsets of a universal set U. Then the ***union*** of set A and B, denoted by $A \cup B$, is the set of all elements $a \in U$ such that $a \in A$ or $a \in B$. It is written as

$$A \cup B = \{a \in U \,|\, a \in A \,\text{or}\; a \in B\}.$$

Intersection of Sets

Definition 2.12. Let A and B be subsets of a universal set U. Then the ***intersection*** of set A and B, denoted by $A \cap B$, is the set of all elements $a \in U$ such that $a \in A$ and $a \in B$. It is written as

$$A \cup B = \{a \in U \,|\, a \in A \,\text{and}\; a \in B\}.$$

Difference of Sets

Definition 2.13. Let A and B be the subsets of universal set U. Then the ***difference*** B minus A or ***relative complement*** of A in B, denoted by $B - A$, is the set of all elements a in U such that $a \in B$ and $a \notin A$.
It is written as

$$B - A = \{a \in U \,|\, a \in B \,\text{and}\; a \notin A\}.$$

Similarly, the difference $A - B$ is the set of all elements $a \in A$ and $a \notin B$.

Complement of Set

Definition 2.14. Let A be the subset of the universal set U. Then ***complement*** of A, denoted by $\bar{A}$, is the set of all the elements a in U such that a is not in A.
It is written as

$$\bar{A} = \{a \in U \mid a \notin A\}.$$

The complement of set A is also denoted as A^C.

Cardinality of a Set

Definition 2.15. The total number of unique elements in the set is called the ***cardinality of the set***.

Example 2.8. The cardinality of set $A = \{a, b, c, d, e\}$ is 5, whereas the cardinality of set $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ is countably infinite.

Venn Diagrams (self study).

Example 2.9. Let

$$A = \{1, 3, 5\}, B = \{2, 3, 5, 7\}.$$

Find $A \cup B, A \cap B, A - B, B - A$.

Solution.

$$A \cup B = \{1, 2, 3, 5, 7\},\ A \cap B = \{3, 5\},\ A - B = \{1\},\ B - A = \{2, 7\}.$$

We note that for sets A and B,
(i) $A \cap B \subseteq A$ and $A \cap B \subseteq B$
(ii) $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

Symmetric Difference of Sets

Definition 2.16. Let A and B be two sets. Then the ***symmetric difference*** between the two sets A and B, denoted by $A \oplus B$ or $A \bigtriangleup B$, is the set containing all the elements that are in A or in B but not in both.
Symbolically,

$$A \oplus B = \{(A \cup B) - (A \cap B)\}.$$

Example 2.10. Let

$$A = \{a, b, c\},\ B = \{c, d, e, f\}.$$

Find $A \oplus B$.

Solution.

$$\begin{aligned} A \oplus B &= (A \cup B) - (A \cap B) \\ &= \{a, b, c, d, e, f\} - \{c\} \\ &= \{a, b, d, e, f\}. \end{aligned}$$

Remark: The symmetric difference of two sets A and B can also be computed as

$$A \oplus B = (A - B) \cup (B - A).$$

In the previous example, we see that $A - B = \{a, b\}$ and $B - A = \{d, e, f\}$. Then it follows that,

$$A \oplus B = (A - B) \cup (B - A) = \{a, b, d, e, f\}.$$

2.2 Algebra of Sets

We now state various laws and identities that sets satisfy when the sets are being operated by union, intersection and complement.

1. **Commutative Laws:**

   $A \cup B = B \cup A, \qquad A \cap B = B \cap A$

2. **Associative Laws:**

   $A \cup (B \cup C) = (A \cup B) \cup C$
   $A \cap (B \cap C) = (A \cap B) \cap C$

3. **Distributive Laws:**

   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. **Idempotent Laws:**

   $A \cup A = A, \qquad A \cap A = A$

5. **Properties of Universal Set:**

   $A \cup U = U, \qquad A \cap U = A$

6. **Absorption Laws:**

   $A \cup (A \cap B) = A, \qquad A \cap (A \cup B) = A$

7. **Complement Law:**

   $A \cap \overline{A} = \emptyset$

8. **Double Complement Law:**

   $\overline{(\overline{A})} = A$

9. **De Morgan's Laws:**

   $\overline{(A \cup B)} = \overline{A} \cap \overline{B}, \qquad \overline{(A \cap B)} = \overline{A} \cup \overline{B}$

10. **Alternate representation for set difference**

    $A - B = A \cap \overline{B}$

Disjoint Sets

Definition 2.17. Two sets A and B are said to be ***disjoint*** if and only if they have no element in common. If A and B are disjoint sets, then

$$A \cap B = \emptyset.$$

Finite Set

Definition 2.18. A set A is said to be ***finite*** if it has n distinct or unique elements, where $n \in \mathbb{N}$. In this case, n is called the ***cardinality*** of A and is denoted by $|A|$.

Example 2.11. $A = \{8, 4, 5, 0, 3\}$ is a finite set, where its cardinality is 5 i.e. $|A| = 5$.

Infinite Set

Definition 2.19. A set that consists of infinite number of different elements or a set that is not finite is called ***infinite*** set.

Example 2.12. A set of integers $\mathbb{Z}$, and a set of natural numbers $\mathbb{N}$ are infinite sets.

★ Addition Principle or Inclusion-Exclusion Principle

If A and B are finite sets, then $A \cup B$ and $A \cap B$ are finite and

$$|A \cup B| = |A| + |A| - |A \cap B|$$

Similarly, if A, B and C are finite sets, then

$$|A \cup B \cup B| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Cartesian Product of Sets

Definition 2.20. Let A and B be two sets. Then the ***cartesian product*** of A and B, denoted by $A \times B$, is defined as the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Thus,

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

If $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$. Hence,

$$A \times \emptyset = \emptyset.$$

If we find the cartesian product of the set A itself, then $A \times A$ can also be denoted by A^2.

The set of elements $(a, a) \in A \times A$ is known as the ***diagonal of*** $A \times A$.

Example 2.13. Let $A = \{a, b\}$, then the cartesian product $A \times A$ is

$$A \times A = A^2 = \{(a, a), (a, b), (b, a), (b, b)\}$$

Example 2.14. Let

$$A = \{1, 2, 3\}, \text{ and } B = \{a, b\}.$$

Then,

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\},$$

and

$$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}.$$

It is important to note that,

$$A \times B \neq B \times A$$

Remark: If A has m elements and B has n elements, then $A \times B$ has mn elements.

2.3 Relations

Definition 2.21. Let A and B be two sets. Then a subset R of $A \times B$ is called a ***relation in*** A ***and*** B.

Given an ordered pair $(a, b) \in A \times B$, a is related to b by R, written as $a\,R\,b$, if and only if $(a, b) \in R$. If they are not related, then we write $a \not R b$ to denote $(a, b) \notin R$.
If $B = A$, the R is called a ***relation on*** A.
The set of first components of pairs in R is called ***relation domain*** of R.
The set of last components of pairs in R is called ***relation range*** of R.
Hence, we have
Relation domain of $R = \{a \mid (a, b) \in R\}$, and
Relation range of $R = \{b \mid (a, b) \in R\}$.

If we denote the domain of R by $D(R)$ and the range of R by $R(R)$, then we have

$$D(R) \subseteq A \text{ and } R(R) \subseteq B.$$

If R is a relation of A on B, then R^{-1}, the relation of B on A is defined by

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}.$$